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# REDUCING DEHN FILLINGS AND $\mathbb{S}^1$ -FACES (On Heegaard Splittings and Dehn surgeries of 3-manifolds, and topics related to them)

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## REDUCING DEHN FILLINGS AND $x$ -FACES

SANGYOP LEE, SEUNGSANG OH<sup>1</sup>, AND MASAKAZU TERAGAITO

**ABSTRACT.** In this paper we investigate the distances between Dehn fillings on a hyperbolic 3-manifold that yield 3-manifolds containing essential small surfaces. We study the situations where one filling creates an essential sphere, and the other creates an essential sphere, annulus or torus.

### 1. INTRODUCTION

Let  $M$  be a compact, connected, orientable 3-manifold with a torus boundary component  $\partial_0 M$ . Let  $\gamma$  be a *slope* on  $\partial_0 M$ , that is, the isotopy class of an essential simple closed curve on  $\partial_0 M$ . The 3-manifold obtained from  $M$  by  $\gamma$ -Dehn filling is defined to be  $M(\gamma) = M \cup V_\gamma$ , where  $V_\gamma$  is a solid torus glued to  $M$  along  $\partial_0 M$  in such a way that  $\gamma$  bounds a meridian disk in  $V_\gamma$ .

By a *small* surface we mean one with non-negative Euler characteristic including non-orientable surfaces. Such surfaces play a special role in the theory of 3-dimensional manifolds. We say that a 3-manifold  $M$  is *hyperbolic* if  $M$  with its boundary tori removed admits a complete hyperbolic structure of finite volume with totally geodesic boundary. Thurston's geometrization theorem for Haken manifolds [Th] asserts that a hyperbolic 3-manifold  $M$  with non-empty boundary contains no essential small surfaces.

If  $M$  is hyperbolic, then the Dehn filling  $M(\gamma)$  is also hyperbolic for all but finitely many slopes [Th], and a good deal of attention has been directed towards obtaining a more precise quantification of this statement. Following [Go2], we say that a 3-manifold is of

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type  $S$ ,  $D$ ,  $A$  or  $T$ , if it contains an essential sphere, disk, annulus or torus, respectively. We denote by  $\Delta(\gamma_1, \gamma_2)$  the distance, or minimal geometric intersection number, between two slopes  $\gamma_1, \gamma_2$  on  $\partial_0 M$ .

Suppose that  $M(\gamma_i)$  for  $i = 1, 2$  contains an essential small surface  $\widehat{F}_i$ . Then we may assume that  $\widehat{F}_i$  meets the attached solid torus  $V_{\gamma_i}$  in a finite collection of meridian disks, and is chosen so that the number of disks  $n_i$  is minimal among all such surfaces in  $M(\gamma_i)$ . Since  $M$  is hyperbolic,  $n_i$  is positive.

In this paper we investigate the distances between two Dehn fillings where one filling creates an essential sphere, and the other creates an essential sphere, annulus or torus. For the first case which is called the reducibility theorem, we give the same result but a simpler proof, and for the other two cases we announce stronger results. The main point is that the generic parts of the proofs of all cases use the same argument.

**Theorem 1.1** (Gordon-Luecke). *Suppose that  $M$  is hyperbolic. If  $M(\gamma_1)$  and  $M(\gamma_2)$  are of type  $S$  then  $\Delta(\gamma_1, \gamma_2) \leq 1$ .*

**Theorem 1.2.** *Suppose that  $M$  is hyperbolic. If  $M(\gamma_1)$  is of type  $S$  and  $M(\gamma_2)$  is of type  $A$  then  $\Delta(\gamma_1, \gamma_2) \leq 1$ , or  $\Delta(\gamma_1, \gamma_2) = 2$  with  $n_2 = 2$ .*

**Theorem 1.3.** *Suppose that  $M$  is hyperbolic. If  $M(\gamma_1)$  is of type  $S$  and  $M(\gamma_2)$  is of type  $T$  then  $\Delta(\gamma_1, \gamma_2) \leq 2$ , or  $\Delta(\gamma_1, \gamma_2) = 3$  with  $n_2 = 2$ .*

## 2. GRAPHS OF SURFACE INTERSECTIONS

Hereafter  $M$  is a hyperbolic 3-manifold with a torus boundary component  $\partial_0 M$ . In this section we describe how essential small surfaces  $\widehat{F}_1$  and  $\widehat{F}_2$ , in  $M(\gamma_1)$  and  $M(\gamma_2)$  respectively, give rise to labelled intersection graphs  $G_i \subset \widehat{F}_i$  for  $i = 1, 2$  in general context.

As in Section 1, let  $\widehat{F}_i$  be an essential small surface in  $M(\gamma_i)$  with  $n_i = |\widehat{F}_i \cap V_{\gamma_i}|$  minimal. Then  $F_i = \widehat{F}_i \cap M$  is a punctured surface properly embedded in  $M$ , each of whose  $n_i$  boundary components has slope  $\gamma_i$ . Recall that  $n_i$  is positive. Note that the minimality of  $n_i$  guarantees that  $F_i$  is incompressible and  $\partial$ -incompressible in  $M$ .

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We use  $i$  and  $j$  to denote 1 or 2, with the convention that, when both appear,  $\{i, j\} = \{1, 2\}$ . By an isotopy of  $F_1$ , we may assume that  $F_1$  intersects  $F_2$  transversely. Let  $G_i$  be the graph in  $\widehat{F}_i$  obtained by taking as the (fat) vertices the disks  $\widehat{F}_i - \text{Int}F_i$  and as edges the arc components of  $F_i \cap F_j$  in  $\widehat{F}_i$ . We number the components of  $\partial F_i$  as  $1, 2, \dots, n_i$  in the order in which they appear on  $\partial_0 M$ . On occasion we will use 0 instead of  $n_i$  in short. This gives a numbering of the vertices of  $G_i$ . Furthermore it induces a labelling of the end points of edges in  $G_j$  in the usual way (see [CGLS]). We can assume that  $G_1$  and  $G_2$  have neither trivial loops nor circle components bounding disks in their punctured surfaces, since all surfaces are incompressible and boundary-incompressible. For the sake of simplicity we will say that  $F_i$  and  $G_i$  are of type  $X_i$  if  $M(\gamma_i)$  is of type  $X_i$ .

The rest of this section will be devoted to several definitions and well known lemmas. Let  $x$  be a label of  $G_i$ . An  $x$ -edge in  $G_i$  is an edge with label  $x$  at one endpoint, and an  $xy$ -edge is an edge with label  $x$  and  $y$  at both endpoints.

An  $x$ -cycle is a cycle of positive  $x$ -edges of  $G_i$  which can be oriented so that the tail of each edge has label  $x$ . A *Scharlemann cycle* is an  $x$ -cycle that bounds a disk face of  $G_i$ , only when  $n_j \geq 2$ . Each edge of a Scharlemann cycle has the same label pair  $\{x, x+1\}$ , so we refer as  $\{x, x+1\}$ -Scharlemann cycle. The number of edges in a Scharlemann cycle,  $\sigma$ , is called the *length* of  $\sigma$ . In particular, a Scharlemann cycle of length two is called an  $S$ -cycle in short.

An *extended  $S$ -cycle* is the quadruple  $\{e_1, e_2, e_3, e_4\}$  of mutually parallel positive edges in succession and  $\{e_2, e_3\}$  form an  $S$ -cycle, only when  $n_j \geq 4$ .

**Lemma 2.1.** *If  $G_j$  is of type  $S$ ,  $A$  or  $T$ , and  $G_i$  contains a Scharlemann cycle, then  $\widehat{F}_j$  must be separating, and so  $n_j$  is even. Furthermore, when  $G_j$  is of type  $A$  or  $T$  (but  $M(\gamma_j)$  is not of type  $S$ ), the edges of the Scharlemann cycle cannot lie in a disk in  $\widehat{F}_j$ .*

*Proof.* Let  $E$  be a disk face bounded by a Scharlemann cycle with labels, say,  $\{1, 2\}$  in  $G_i$ . Let  $V_{12}$  be the 1-handle cut from  $V_{\gamma_j}$  by the vertices 1 and 2 of  $G_j$ . Then tubing  $\widehat{F}_j$  along  $\partial V_{12}$  and compressing along  $E$  gives such a new  $\widehat{F}_j$  in  $M(\gamma_j)$  that intersects  $V_{\gamma_j}$  fewer times than the old one.

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Note that if  $\widehat{F}_j$  is non-separating, then the new one is also non-separating, and so essential. Furthermore, if the edges of the Scharlemann cycle lie in a disk  $D$  in an annulus or torus  $\widehat{F}_j$ , then  $\text{nhd}(D \cup V_{12} \cup E)$  is a once punctured lens space. The incompressibility of  $\widehat{F}_j$  means that  $M(\gamma_j)$  must be reducible.  $\square$

**Lemma 2.2.** *If  $G_j$  is of type  $S$  or  $A$ , then  $G_i$  cannot contain Scharlemann cycles on distinct label pairs.*

*Proof.* Without loss of generality assume that  $\sigma_1$  is a Scharlemann cycle with label pair  $\{1, 2\}$  bounding a face  $E$  in  $G_i$ .

First, assume that  $G_j$  is of type  $S$ . Consider a punctured lens space  $L = \text{nhd}((\widehat{F}_j - \text{Int}D) \cup V_{12} \cup E)$  where  $D$  is a disk in  $\widehat{F}_j$  separated by the edges of  $\sigma_1$ . Since  $\partial L$  is a reducing sphere in  $M(\gamma_j)$ ,  $2|(\widehat{F}_j - \text{Int}D) \cap V_{\gamma_j}| - 2 = |\partial L \cap V_{\gamma_j}| \geq n_j$  by the minimality of  $n_j$ . Thus  $|\text{Int}D \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$ , i.e. the interior of  $D$  contains at most  $\frac{n_j}{2} - 1$  vertices of  $G_j$ .

Suppose there is another Scharlemann cycle  $\sigma_2$  with distinct label pair. Since the edges of  $\sigma_2$  lie in the same disk component of  $\widehat{F}_j$  separated by the edges of  $\sigma_1$ , the union of the edges of  $\sigma_2$  and the corresponding two vertices should be contained in a disk containing at most  $\frac{n_j}{2}$  vertices (note that  $\sigma_1$  and  $\sigma_2$  can share one label). This implies that the other  $\frac{n_j}{2}$  vertices of  $G_j$  lie in the same disk component of  $\widehat{F}_j$  separated by the edges of  $\sigma_2$ , contradicting the same argument of the preceding paragraph.

Next, assume that  $G_j$  is of type  $A$  (but  $M(\gamma_j)$  is not of type  $S$ ). We give a brief of the proof of [Wu3, Lemma 5.4(2)]. By Lemma 2.1 the union of the edges of  $\sigma_1$  and the corresponding two vertices cannot lie in a disk, so cuts  $\widehat{F}_j$  into two annuli  $A_1, A_2$  and some disk components. Consider the manifold  $Y = \text{nhd}((\widehat{F}_j - \text{Int}A_1) \cup V_{12} \cup E)$ . Claim in Wu's proof guarantees that the frontier  $Q$  of  $Y$  is an essential annulus in  $M(\gamma_j)$ . Then  $2|(\widehat{F}_j - \text{Int}A_1) \cap V_{\gamma_j}| - 2 = |Q \cap V_{\gamma_j}| \geq n_j$  by the minimality of  $n_j$ . Thus  $|\text{Int}A_1 \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$ .

Suppose there is another Scharlemann cycle  $\sigma_2$  with distinct label pair. Since the edges of  $\sigma_2$  cannot lie in a disk, we may assume that these edges are contained in the annulus  $A_1$ . Like before, this implies that one (not contained in  $A_1$ ) of two annuli in  $\widehat{F}_j$  separated by

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the edges of  $\sigma_2$  contains at least  $\frac{n_j}{2}$  vertices of  $G_j$  in its interior, contradicting the previous argument.  $\square$

**Lemma 2.3.** *If  $G_j$  is of type  $S$ ,  $A$  or  $T$ , then  $G_i$  cannot contain an extended  $S$ -cycle.*

*Proof.* All three cases follow from [Wu1, Lemma 2.3], [Wu3, Lemma 5.4(3)] and [BZ, Lemma 2.10], respectively.

In fact one could avoid these reference to [Wu1, Wu3] for the cases of type  $S$  and  $A$ . Note that an extended  $S$ -cycle is the simplest one in  $x$ -faces (defined in Section 3) where  $x$  is not a label of a Scharlemann cycle. Then these two cases follow immediately from Theorems 3.4 and 3.5.  $\square$

The *reduced graph*  $\overline{G}_i$  of  $G_i$  is defined to be the graph obtained from  $G_i$  by amalgamating each family of parallel edges into a single edge.

3.  $x$ -FACE

In this section we assume that  $G_j$  is of type  $S$  or  $A$ . By Lemma 2.2 we may say that  $G_i$  contains only 12-Scharlemann cycles.

A disk face of the subgraph of  $G_i$  consisting of all the vertices and positive  $x$ -edges of  $G_i$  is called an  $x$ -face. Remark that the boundary of an  $x$ -face  $D$  may be not a circle, that is,  $\partial D$  may contain a double edge, and more than two edges of  $\partial D$  may be incident to a vertex on  $\partial D$  (see Figure 2.1 in [HM]). A cycle in  $G_i$  is a *two-cornered cycle* if it is the boundary of a face containing only 01-corners, 23-corners and positive edges, and additionally it contains at least one edge of a 12-Scharlemann cycle. Recall that 0 denotes  $n_j$ . A two-cornered cycle must contain both corners and a 03-edge because  $G_i$  has only 12-Scharlemann cycles. A *cluster*  $C$  is a connected subgraph of  $G_i$  satisfying that

- (i)  $C$  consists of 12-Scharlemann cycles and two-cornered cycles,
- (ii) every 12-edge of  $C$  belongs to both a Scharlemann cycle and a two-cornered cycle,
- and
- (iii)  $C$  contains no cut vertex.

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The notions of a two-cornered cycle and a cluster were used firstly in [Ho]. In the paper Hoffman showed that the disk bounded by a great  $x$ -cycle contains a pair of specific two-cornered cycles, called a seemly pair, and it can be used to find a new essential sphere meeting the attached solid torus in fewer times, leading to a contradiction.

**Lemma 3.1.** *Suppose that  $n_j \geq 3$ . An  $x$ -face,  $x \neq 1, 2$ , in  $G_i$  contains a cluster  $C$ .*

*Proof.* Let  $\Gamma_D$  be the subgraph of  $G_i$  in an  $x$ -face  $D$ . There is a possibility that  $\partial D$  is not a circle as mentioned before. Since we will find a pair of two-cornered cycles within  $D$ , we can cut formally the graph  $G_i \cap D$  along double edges of  $\partial D$  and at vertices to which more than two edges of  $\partial D$  are incident to so that  $\partial D$  is deformed into a circle. (See also Figure 5.1 in [HM].) Thus we may assume that  $\partial D$  is a circle and  $\Gamma_D$  has no vertex in the interior of  $D$ . We may assume that the labels appear in anticlockwise order around the boundary of each vertex.

Suppose that  $D$  has a diagonal edge  $d$  with a label pair  $\{a, b\}$ , which are not of 12-Scharlemann cycles, as in Figure 1(a). Note that these labels must differ from  $x$ . Assume without loss of generality that  $b < a < x$ , i.e. three labels  $b, a, x$  appear in anticlockwise order in usual sence. Formally construct a new  $x$ -face  $D'$  as follows. Keep all corners and edges of  $\Gamma_D$  to the right of  $d$  (when  $d$  is directed from  $a$  to  $b$ ), discard all corners and edges to the left of  $d$ , and then insert additional edges to the left of  $d$ , and parallel to  $d$ , until you first reach label  $x$  at one end of this parallel family of edges, as in Figure 1(b). In particular, these additional edges contain no edges of two-cornered cycles or Scharlemann cycles of the graph on the new  $x$ -face  $D'$ .

Repeat the above process for every diagonal edges which are not of 12-Scharlemann cycles, then we get a new  $x$ -face  $E$  and a graph  $\Gamma_E$  in  $E$  so that all diagonal edges are of 12-Scharlemann cycles and all (and only) boundary edges are  $x$ -edges. Furthermore the additional edges contain no edges of Scharlemann cycles or two-cornered cycles of  $\Gamma_E$ . Remark that an  $xx$ -edge can appear on the boundary of the graph because we cannot guarantee that  $\widehat{F}_j$  is separating.

**Claim 3.2.**  $\Gamma_E$  contains a 12-Scharlemann cycle, so does  $\Gamma_D$ .

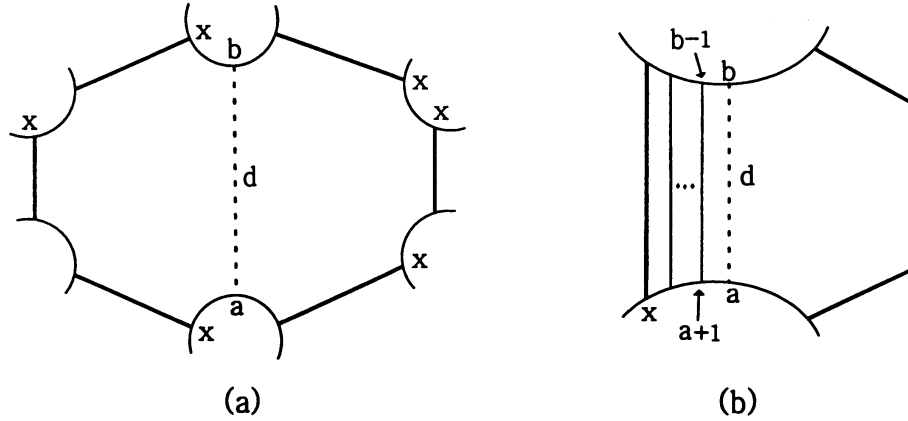
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FIGURE 1. Split along a diagonal edge

*Proof.* Assume that  $\Gamma_E$  contains no 12-Scharlemann cycles, and so no diagonal edges. We show first that if for some vertex  $v$  of  $\Gamma_E$  two boundary edges are incident to  $v$  with label  $x$ , then these should be  $xx$ -edges. For, in  $\Gamma_E$  at least  $n_j + 1$  edges are incident to  $v$ . If  $n_j$  is even, more than  $\frac{n_j}{2}$  mutually parallel edges are incident to  $v$ , and so one of these edges should be a  $yy$ -edge,  $y \neq x$ , (recall that  $\Gamma_E$  cannot contain a Scharlemann cycle), a contradiction. If  $n_j$  is odd, two families of  $\frac{n_j+1}{2}$  mutually parallel edges are incident to  $v$ , and the boundary edge of each family should be an  $xx$ -edge by the same reason above.

Consider the cycle  $\sigma$  consisting of boundary  $x$ -edges of  $\Gamma_E$ . Assume that  $\sigma$  has an  $x$ -edge which is not an  $xx$ -edge. So only one end has label  $x$  at  $v_1$ , say. By the fact we just proved, another  $x$ -edge incident to  $v_1$  does not have label  $x$  at  $v_1$ . Thus this edge has label  $x$  at the other end  $v_2$ , say. After repeating this process, we are led to show that  $\sigma$  is a great  $x$ -cycle in the terminology of [CGLS]. (If all boundary edges are  $xx$ -edges, still we have a great  $x$ -cycle.) By the same argument in the proof of Lemma 2.6.2 of [CGLS],  $\Gamma_E$  contains a Scharlemann cycle, a contradiction.  $\square$

This means that  $\widehat{F}_j$  must be separating and  $n_j$  is even by Lemma 2.1. The parity rule guarantees that each edge of  $\Gamma_E$  connects vertices with one label even and the other label odd, and so there is no  $xx$ -edges.



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Any 12-edge of a Scharlemann cycle does not belong to  $\partial\Gamma_E$ . Consider the face  $E_1$  of  $\Gamma_E$  adjacent to the 12-edge which does not bound the Scharlemann cycle. It is possible that  $E_1$  contains more than one 12-edges of Scharlemann cycles. Let  $\{a_k, a_k + 1\}$ ,  $k = 1, \dots, n$ , be the consecutive label pairs of the corners between two consecutive 12-edges of Scharlemann cycles when runs clockwise around  $\partial E_1$ . Note that  $a_1 = 2$  and  $a_n = 0$ .

Assume for contradiction that  $\partial E_1$  is not a two-cornered cycle. Since some  $a_k$  then is neither 0 nor 2, there are indices  $l$  and  $m$  so that  $a_k = 0$  or 2 when  $1 \leq k < l$  or  $k = m$ , and  $a_k \neq 0, 2$  when  $l \leq k < m$ .

Consider the edges of the parallelism class containing each  $\{a_{k-1} + 1, a_k\}$ -edge for  $l \leq k \leq m$ . Since there is no Scharlemann cycles among these edges, one finds that  $x \leq a_k < a_{k-1} + 1 \leq x$ , or  $x \leq a_k \leq a_{k-1} < x$ . And so  $x \leq a_m \leq a_{m-1} \leq \dots \leq a_l \leq a_{l-1} < x$ . This is impossible because  $a_{l-1}, a_m = 0$  or 2 and all  $a_k$ 's are even by the parity rule. Thus  $\partial E_1$  is a two-cornered cycle. Let  $C$  be the union of all the Scharlemann cycles and all the two-cornered cycles adjacent to each 12-edges of the Scharlemann cycles. After cutting along cut vertices, a connected component of  $C$  is then a desired cluster in  $\Gamma_E$  and so in  $\Gamma_D$ .  $\square$

Let  $\tilde{F}_j$  be the twice-punctured sphere obtained from  $\hat{F}_j$  by deleting two fat vertices 1 and 2 (if  $G_j$  is of type A, then use  $\hat{F}_j$  after capping off two boundary circles by disks). The family of all 12-edges of a Scharlemann cycle in the cluster  $C$  separates  $\tilde{F}_j$  into disks, and one of those disks contains both vertices 0 and 3 of  $G_j$  because of the existence of 03-edges in  $C$ . The two 12-edges bounding such a disk is called *good edges* of  $C$ . Thus each Scharlemann cycle in  $C$  has exactly two good edges.

Let  $\Lambda$  be the maximal dual graph of  $C$  whose vertices are dual to Scharlemann cycles and two-cornered cycles containing good edges, and edges are dual to good edges of  $C$  as depicted in Figure 2. Thus in  $\Lambda$ , a vertex dual to a Scharlemann cycle has valency 2, and a vertex dual to a two-cornered cycle has valency the number of good edges of the two-cornered cycle. Furthermore  $\Lambda$  is a tree according to the construction of  $C$ . This implies that each 12-edge, which is not a good one, of two-cornered cycles related to vertices of

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$\Lambda$  contributes to the number of components of  $\Lambda$  by adding 1. Consequently there is a component  $\Lambda_g$  of  $\Lambda$  so that all 12-edges of its dual two-cornered cycles are good.

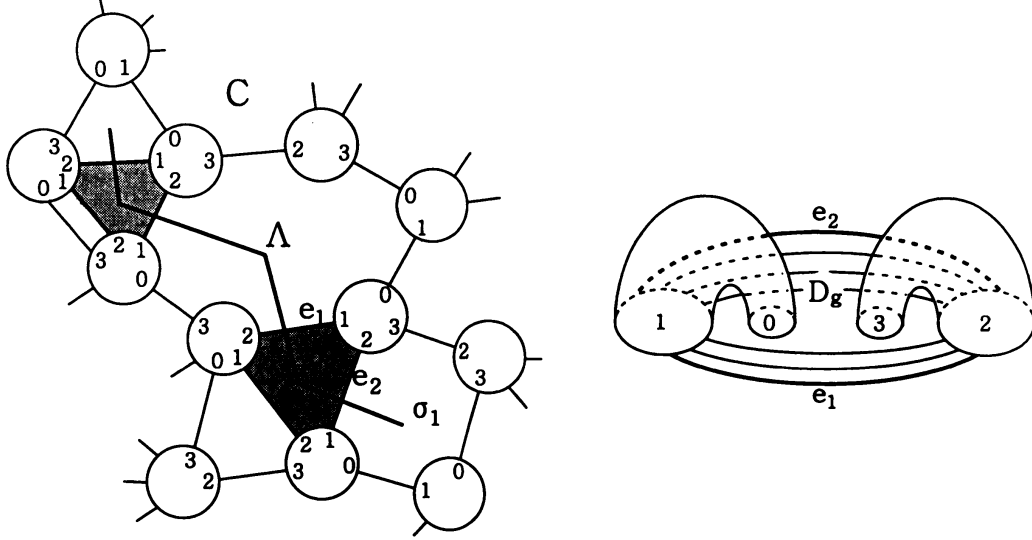


FIGURE 2. A cluster and a seemly pair

Hereafter, we consider the subgraph  $C_g$  of  $C$ , dual to  $\Lambda_g$ . Say,  $C_g$  contains  $n$  Scharlemann cycles, and so  $2n$  good edges and  $n + 1$  two-cornered cycles. A two-cornered cycle dual to an end vertex of the tree  $\Lambda_g$  has only one good edge. Choose one  $e_1$  of the nearest edges to vertex 0 (or 3) among them, i.e. there is no such good edges between  $e_1$  and vertex 0 in  $\tilde{F}_j$ . Let  $\sigma_g$  and  $\sigma_1$  be the Scharlemann cycle and two-cornered cycle adjacent to  $e_1$  respectively. Then  $\sigma_g$  has another good edge  $e_2$ , and  $e_1$  and  $e_2$  bound a disk  $D_g$  containing 0 and 3 in  $\tilde{F}_j$ . Note that all Scharlemann cycles are parallel on a torus obtained from  $\tilde{F}_j$  by attaching an annulus  $\partial V_{12}$ . Thus exactly  $n - 1$  out of  $2n$  good edges are not contained in  $D_g$ . Therefore we have another two-cornered cycle  $\sigma_2$  all of whose 12-edges lie in  $D_g$ . Consequently all edges (consisting of 01-edges, 12-edges, 23-edges and 03-edges) of  $\sigma_1$  and  $\sigma_2$  lie in  $D_g$ . Furthermore if  $\sigma_2$  has only one good 12-edge, then the two good edges of  $\sigma_1$  and  $\sigma_2$  lie on different sides of the vertices 0 and 3 in  $D_g$ . Such  $\sigma_1, \sigma_2$  are called a *seemly pair*. Then we can say;

**Lemma 3.3.** *There is a seemly pair of two-cornered cycles in  $C$ .*

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From now we apply the argument in [Ho, Section 6] to get the following two theorems.

**Theorem 3.4.** *If  $G_j$  is of type  $S$  with  $n_j \geq 3$ , then  $G_i$  cannot contain an  $x$ -face for  $x \neq 1$  or 2.*

*Proof.* Suppose that  $G_i$  contains such an  $x$ -face. We continue the preceding argument. Recall that  $|\text{Int} D_g \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$  as in the proof of Lemma 2.2. Let  $X'_D = \text{nhd}(D_g \cup V_{01} \cup V_{23})$  and  $X'_F = \text{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23})$ . Then  $X'_D$  is a genus two handlebody and  $X'_F$  is a once-punctured genus two handlebody. The genus two torus component of  $\partial X'_F$  is referred to as the outer boundary of  $X'_F$ . Let  $E_i$  be the face bounded by the two-cornered cycle  $\sigma_i$  for  $i = 1, 2$ . The point is that all edges of  $\sigma_i$  are contained in  $D_g$ . Let  $X_D = X'_D \cup \text{nhd}(E_1) \cup \text{nhd}(E_2)$  and  $X_F = X'_F \cup \text{nhd}(E_1) \cup \text{nhd}(E_2)$  as in Figure 3. Since  $\sigma_1$  is non-separating on  $\partial X'_D$  and  $\partial X'_F$ , both  $\partial X_D$  and the outer components of  $\partial X_F$  are either a 2-sphere or the disjoint union of a 2-sphere and a torus, simultaneously. Note that the latter case occurs only when  $\sigma_1$  and  $\sigma_2$  are parallel.

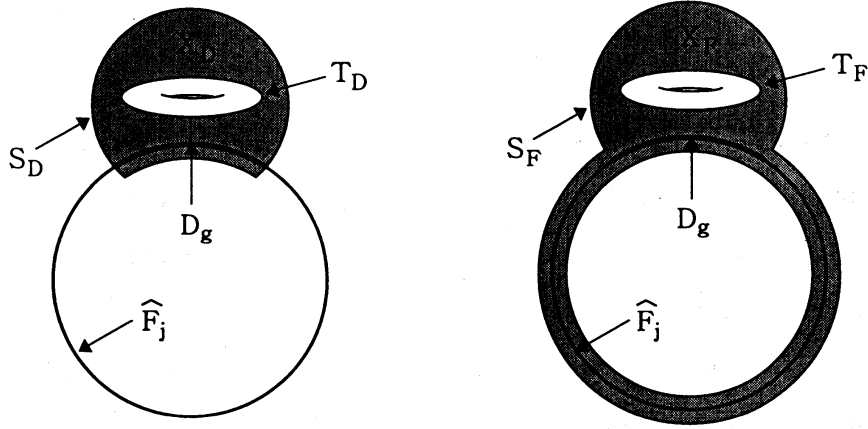


FIGURE 3.  $X_D$  and  $X_F$

First, assume that  $\partial X_D$  is a 2-sphere  $S_D$ , and the outer component of  $\partial X_F$  is a 2-sphere  $S_F$ . If  $X_D$  is not a 3-ball, then  $S_D$  is a reducing sphere. By the previous remark  $|S_D \cap V_{\gamma_j}| = 2|D_1 \cap V_{\gamma_j}| \leq n_j - 2$ , contradicting the minimality of  $n_j$ . Thus it should be a 3-ball, and so  $X_F$  is homeomorphic to  $S^2 \times I$ . Thus  $S_F$  is isotopic to  $\widehat{F}_j$ , contradicting the minimality of  $n_j$  again.

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Next, assume that  $\partial X_D$  is the disjoint union of a 2-sphere and a torus,  $S_D \cup T_D$  and the outer components of  $\partial X_F$  are also the disjoint union of a 2-sphere and a torus,  $S_F \cup T_F$ . Recall that this case occurs whenever  $\sigma_1$  and  $\sigma_2$  cobound an annulus in  $\partial X'_D$  and  $\partial X'_F$ . This is possible only when  $\sigma_2$  corresponds to an end vertex of  $\Lambda_g$  with the same number of 01-corners and 23-corners as that of  $\sigma_1$ . Let  $D'$  be the intersection of  $\partial X_D$  and the inner sphere component of  $\partial X_F$ . By the choice of the seemly pair, this annulus in  $\partial X'_D$  contains  $D'$ , so does  $S_D$ . Similarly  $S_F$  contains a pushoff of  $\tilde{F}_j - D_g$ .

If  $S_D$  is non-separating in  $M(\gamma_j)$ , then it is a reducing sphere with  $|S_D \cap V_{\gamma_j}| \leq n_j - 2$ , contradicting. Thus  $S_D$ , and hence  $T_D$  are separating in  $M(\gamma_j)$ . Let  $X''_D$  be the manifold bounded by  $S_D$  containing  $T_D$  in  $M(\gamma_j)$ . If  $X''_D$  is not a 3-ball, then  $S_D$  is a reducing sphere with less intersection with  $V_{\gamma_j}$ . Thus it should be a 3-ball, and so  $S_F$  is isotopic to  $\hat{F}_j$ , contradicting again. This completes the proof.  $\square$

**Theorem 3.5.** *If  $G_j$  is of type A with  $n_j \geq 3$ , then  $G_i$  cannot contain an  $x$ -face for  $x \neq 1$  or 2.*

*Proof.* Suppose that  $G_i$  contains such an  $x$ -face. By Theorem 3.4  $M(\gamma_j)$  is irreducible. Again we continue the argument stated before Lemma 3.3. We distinguish three cases;

First, suppose  $D_g$  contains  $\partial \hat{F}_j$ . Then the edges of the Scharlemann cycle  $\sigma_g$  lie in a disk in  $\hat{F}_j$ , contradicting Lemma 2.1.

Second, suppose  $D_g$  does not contain any component of  $\partial \hat{F}_j$ . Then  $D_g$  is contained in  $\hat{F}_j$ . The proof is similar to that of the preceding theorem. Let  $X_D = \text{nhd}(D_g \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$  and  $X_F = \text{nhd}(\hat{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$ . Then  $\partial X_D$  is either a 2-sphere or the disjoint union of a 2-sphere and a torus, and the (similarly defined) outer components of the frontier of  $X_S$  are either an annulus or (by the choice of the seemly pair) the disjoint union of an annulus and a torus. The latter two cases occur only when  $\sigma_1$  and  $\sigma_2$  are parallel as previous. Since  $M(\gamma_j)$  is irreducible, the 2-sphere component of  $\partial X_D$  should bound a 3-ball. Thus the annular component of the outer components of the frontier of  $X_F$  is isotopic to  $\hat{F}_j$ , contradicting the minimality of  $n_j$ .

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Finally suppose  $D_g$  contains exactly one component of  $\partial\widehat{F}_j$ . Let  $A_1$  be the annulus  $D_g \cap F_j$ . Note that  $|\text{Int}A_1 \cap V_{\gamma_j}| \leq \frac{n_j}{2} - 1$  as in the proof of Lemma 2.2. Let  $X_A = \text{nhd}(A_1 \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$  and  $X_F = \text{nhd}(\widehat{F}_j \cup V_{01} \cup V_{23} \cup E_1 \cup E_2)$  again. Then  $\partial X_A$ , and also the outer components (which is not a pushoff of  $\widehat{F}_j$  on the other side of attached handles  $V_{01}$  or  $V_{23}$ ) of the frontier of  $X_F$ , is either an annulus or the disjoint union of an annulus and a torus. Let  $A_A$  and  $A_F$  be the corresponding annular components of the boundaries of  $X_A$  and  $X_F$  respectively. If  $A_A$  is essential in  $M(\gamma_j)$ , then  $|A_A \cap V_{\gamma_j}| = 2|\text{Int}A_1 \cap V_{\gamma_j}| \leq n_j - 2$ , contradicting the minimality of  $n_j$ . Remark that  $A_A$  is incompressible because the central curve of  $A_A$  is isotopic to the central curve of  $\widehat{F}_j$  and  $\widehat{F}_j$  is incompressible. Thus  $A_A$  should be  $\partial$ -compressible, i.e. one of the manifolds  $X_A$  and  $M(\gamma_j) - \text{Int}X_A$  contains a  $\partial$ -compressing disk. Since  $M(\gamma_j)$  is irreducible and  $\partial$ -irreducible, the manifold with the  $\partial$ -compressing disk is a solid torus with  $A_A$  as a longitudinal annulus. Through this manifold one can isotope  $\widehat{F}_j$  to  $A_F$  which intersects  $V_{\gamma_j}$  fewer times than  $\widehat{F}_j$ .  $\square$

Remark that an extended  $S$ -cycle is the boundary cycle of the simplest  $x$ -face. Thus two theorems above guarantee that if  $G_j$  is of type  $S$  or  $A$ , then  $G_i$  cannot contain an extended  $S$ -cycle.

#### 4. PROOFS

In this section we prove the main theorems. Assume that  $M(\gamma_1)$  is of type  $S$ ,  $M(\gamma_2)$  is one of three types, and  $\Delta \geq 2$ . Note that  $n_1 \geq 3$  because  $M$  does not contain essential small surfaces.

**Proposition 4.1.**  *$G_1$  contains a connected subgraph  $\Lambda$  so that it has a disk support  $D$  in  $\widehat{F}_1$  such that  $D \cap G_1^+ = \Lambda$ , and each boundary vertex, except at most one vertex  $y_0$ , has degree at least  $(\Delta - 1)n_2 + \chi(\widehat{F}_2)$  in  $\Lambda$ .*

*Proof.* If a vertex  $x$  of  $G_1$  has more than  $n_2 - \chi(\widehat{F}_2)$  negative edges, then  $G_2$  contains more than  $n_2 - \chi(\widehat{F}_2)$  positive  $x$ -edges by the parity rule. Thus the subgraph  $\Gamma_x$  of  $G_2$  consisting of all vertices and positive  $x$ -edges of  $G_2$  has  $n_2$  vertices and more than  $n_2 - \chi(\widehat{F}_2)$  edges. Then an Euler characteristic calculation shows that  $\Gamma_x$  contains a disk face, that is, an

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$x$ -face. Thus by Theorem 3.4 each vertex  $x \neq 1, 2$ , the labels of Scharlemann cycles of  $G_2$  if it exist, has at least  $(\Delta - 1)n_2 + \chi(\widehat{F}_2)$  positive edges.

Let  $G_1^+$  denote the subgraph of  $G_1$  consisting of all vertices and positive edges of  $G_1$ . Let  $\Lambda'$  be an extremal component of  $G_1^+$ . That is,  $\Lambda'$  is a component of  $G_1^+$  having a disk support  $D$  such that  $D \cap G_1^+ = \Lambda'$ . In  $\Lambda'$ , a block  $\Lambda$  with at most one cut vertex is called an *extremal block*. If  $\Lambda'$  has no cut vertex, then  $\Lambda'$  itself is an extremal block, and if  $\Lambda'$  has a cut vertex, then it has at least two extremal blocks.

Furthermore, if  $G_2$  contains a 12-Scharlemann cycle, then  $\widehat{F}_1$  is separating, and so  $G_1^+$  is disconnected. Then there is an extremal component of  $G_1^+$  which contains at most one of vertices 1 and 2. Now choose an extremal block so that it contains at most one such vertex or a cut vertex which is called a *ghost vertex*  $y_0$ . Then  $\Lambda$  is the desired subgraph.  $\square$

*Proof of Theorem 1.1.*  $M(\gamma_2)$  is of type  $S$ . By Proposition 4.1,  $G_1$  contains an extremal block  $\Lambda$ , each of whose boundary vertex except  $y_0$  has at least  $n_2 + 2$  consecutive edge endpoints, and so has all different  $n_2$  labels.

If there is no ghost  $y_0$ , choose any label  $x'$  but 1 and 2, the labels of Scharlemann cycles of  $G_1$ . Or if such  $y_0$  exists and it has more than two edges incident there, then choose  $x'$  among the labels of  $y_0$  except 1 and 2. Let  $\Gamma_{x'}$  be the subgraph of  $\Lambda$  consisting of all vertices and  $x'$ -edges. Then in  $\Gamma_{x'}$  the number of edges cannot be less than the number of vertices. Again an Euler characteristic calculation of  $\Gamma_{x'}$  on the disk guarantees the existence of  $x'$ -face,  $x' \neq 1, 2$ , contradicting Theorem 3.4. For the remaining case, if  $y_0$  has only two edges incident there, then delete  $y_0$  and the two edges from  $\Lambda$ . Since all labels still appear on each vertex of this new  $\Lambda$ , we can proceed the same argument above to get a contradiction.  $\square$

*Proof of Theorem 1.2.*  $M(\gamma_2)$  is of type  $A$ . For the case  $n_2 = 1$ ,  $G_2$  can have only positive edges. Thus it contains an  $x$ -face,  $x \neq 1, 2$ , contradicting Theorem 3.4.

We assume therefore that  $n_2 \geq 3$ . Again by Proposition 4.1,  $G_1$  contains an extremal block  $\Lambda$ , each of whose boundary vertex except  $y_0$  has at least  $n_2$  consecutive edge endpoints, i.e. all different  $n_2$  labels. Then it contains a 12-Scharlemann cycle as in Claim

3.2. By Lemma 2.1  $n_2$  is even and this Scharlemann cycle divides the annulus  $\widehat{F}_2$  into two disjoint annuli  $A_1$  and  $A_2$ . Note that there must be  $y_0$  with only two edges incident to at labels 1 and 2 in  $\Lambda$ , and there is no interior vertex of  $\Lambda$ . Otherwise, an Euler characteristic calculation shows that  $\Lambda$  contains an  $x$ -face for  $x \neq 1, 2$ , so does  $G_1$ . But this contradicts Theorem 3.5. Furthermore  $n_2 = 4$  and the two edges incident to  $y_0$  are indeed 14, 23-edges, otherwise  $\Lambda$  after deleting  $y_0$  and these two edges also contains an  $x$ -face for  $x \neq 1, 2$ , contradicting as previous. By the same reason, each boundary vertices of  $\Lambda$  except  $y_0$  has exactly four edges incident to. From the facts above one can conclude that  $\Lambda$  has a 34-edge. That means vertices 3 and 4 lie on the same annulus  $A_1$  or  $A_2$ .

But, one vertex, say  $x'$ , of  $\Lambda$  has four consecutive negative edges attached in  $G_1$ . By the parity rule these four edges produce two disjoint  $x'$ -cycles in  $G_2$ , each of which can not lie in a disk in  $\widehat{F}_2$ . Furthermore one connects vertices 1 and 3, and the other connects vertices 2 and 4. Thus vertices 3 and 4 can not lie on the same annulus  $A_1$  or  $A_2$ , contradicting.

For the case  $n_2 = 1$ , done by [Wu3, Theorem 5.1].  $\square$

*Proof of Theorem 1.3.*  $M(\gamma_2)$  is of type  $T$ . Assume that  $\Delta \geq 3$ . For the case  $n_2 = 1$ ,  $G_2$  can have only positive edges. Thus it contains an  $x$ -face,  $x \neq 1, 2$ , contradicting Theorem 3.4.

We now assume that  $n_2 \geq 3$ . By Proposition 4.1,  $G_1$  contains an extremal block  $\Lambda$ , each of whose boundary vertex except  $y_0$  has at least  $2n_2$  consecutive edge endpoints.

There is a label  $x$  of  $G_1$  which is not of  $S$ -cycles of  $G_1$ . For, if not, all labels of  $G_1$  are of  $S$ -cycles. Then by Lemma 2.1,  $n_2 \geq 4$ , and so there are two  $S$ -cycles  $\sigma_1$  and  $\sigma_2$  with disjoint label pairs. Let  $\{\alpha_k, \alpha_k + 1\}$  be the labels of  $\sigma_k$ , and let  $E_k$  be the face of  $G_1$  bounded by  $\sigma_k$ ,  $k = 1, 2$ . Then shrinking  $V_{\alpha_k, \alpha_k + 1}$  to its core in  $V_{\alpha_k, \alpha_k + 1} \cup E_k$  gives a Möbius band  $B_k$  such that  $\partial B_k$  is the loop on  $\widehat{F}_2$  formed by the edges of  $\sigma_k$ . By Lemma 2.1,  $\partial B_1$  is isotopic to  $\partial B_2$ . Then taking the union of  $B_1$  and  $B_2$  would give a Klein bottle in  $M(\gamma_2)$ . This is impossible because of [LOT].

Consider the subgraph  $\Lambda^x$  consisting of all vertices and  $x$ -edges of  $\Lambda$ . Since every boundary vertex of  $\Lambda$ , except  $y_0$ , has degree at least  $2n_j$ , it has at least two edges attached with

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label  $x$ . Note that  $\Lambda^x$  may be disconnected. Choose an extremal block  $\Lambda'$  of  $\Lambda^x$  (in a disk support  $D$ ). Let  $v, e$  and  $f$  be the numbers of vertices, edges, and disk faces of  $\Lambda'$ , respectively. Also let  $v_i, v_\partial$  and  $v_g$  be the numbers of interior vertices, boundary vertices and ghost vertices. Hence  $v = v_i + v_\partial$  and  $v_g = 0$  or  $1$ .

Because of Lemma 2.3, each face of  $\Lambda'$  is a disk with at least 3 sides. Thus we have  $3f + v_\partial \leq 2e$ . Since it has only disk faces, combined with  $v - e + f = \chi(D) = 1$ , we get  $e \leq 3v_i + 2v_\partial - 3$ . On the other hand we have  $2(v_\partial - v_g) + \Delta v_i \leq e$  because each boundary vertex of  $\Lambda'$ , except  $y_0$ , has at least two edges attached with label  $x$ . These two inequalities give us that  $3 \leq 2v_g$ , a contradiction.

For the case  $n_2 = 2$ , done by [Oh, Theorem 1.1] and [Wu2, Theorem 1]. □

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